Note on the Demkov-Ostrovsky nodeless sector

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Summary. - We briefly tackle the following concepts in the Demkov-Ostrovsky (DO) nodeless sector: (i) orbital impedance, (ii) orbital capacity, (iii) closeness to reflectionlessness. Moreover, using previous supersymmetric results for the DO problem, a strictly isospectral effect in the DO orbital impedances is discussed and explicit plots are displayed for the Maxwell fisheye lens. This effect, though rather small, is general, that is, it may apply to any focusing structure.

PACS 03.65 - Quantum mechanics

PACS 11.30 - Supersymmetry

Because of one of the optical-mechanical analogies, the DO problem can be stated as a Schrödinger radial (half line) equation at zero energy with the focusing potential $V_{\kappa}(\rho) = -w/\rho^2[\rho^{-\kappa}+\rho^{\kappa}]^2$, where $w > 0$ is a (Sturmian type) coupling constant, ρ is a scaled radial variable (r/R), and κ (> 0) is the Lenz-Demkov-Ostrovsky parameter, which is unity for the Maxwell fish eye (MFE) lens, and one half for an atomic aufbau model [1]. In previous works [2], a supersymmetric, Witten approach of the DO problem in the nodeless radial sector $n = l + 1$ (1s, 2p, 3d ...) has been put forward. Moreover, the strictly isospectral double Darboux method has been worked out with the interesting result that a one-parameter family of Maxwell lenses having the same optical scattering properties in the nodeless sector might exist [3].

In this note, still within the same nodeless sector, we will refer to the concepts of DO orbital impedance and orbital capacity. Moreover, we mention a problem that we call "closeness to reflectionlessness". The strictly isospectral effect [3] is graphically displayed using the language of impedances that may have some advantages not only in technological applications.

(i) Orbital impedance: Martin and Sabatier showed in a general context that a localized,

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strictly positive zero energy solution (zero mode) of a Schrödinger equation can be interpreted as an inverse impedance [4]. If we claim that the inverses of the DO radial factors in the nodeless sector, i.e., $f_{\kappa,l}^{-1} \equiv Z_{\kappa,l} = \rho^{-(l+1)}(1+\rho^{2\kappa})^{(2l+1)/2\kappa}$ are orbital "impedances", then these DO impedances are singular ($\propto 1/\rho^{l+1}$) at the center of the DO spheres, and also increase sharply beginning at an outside distance of about one diameter, with no relevant structure. On the other hand, we have found that the strictly isospectral radial factors obtained by the double Darboux method [3] are more interesting, precisely because they have a more intricate spatial behavior with the isospectral Darboux parameter playing an important role in the problem. Thus, it may be indeed helpful to consider the inverses of the DO isospectral radial factors as isospectral orbital impedances, having the form $Z_{iso} \equiv f_{\kappa,l}^{-1}(I_{\kappa;l}+\lambda)$, where $I_{\kappa;l} = \int_0^{\rho} f_{\kappa,l}^2(\rho^{'})d\rho^{'}$ and $\lambda \in (0,\infty)$ is the isospectral family parameter, i.e., mathematically speaking, the Riccati integration constant. Z_{iso} is also singular at the DO center. Therefore, in order to show in a clear-cut way the strictly isospectral effect for the MFE lens, we did plots of the ratio $\mathcal{R}_l(\rho) = Z_{iso}/Z_{1,l} = (I_{1,l}(\rho)+\lambda)$ for the partial waves from $l=1$ to $l=3$ and for $\lambda = 0.01,\ 0.1,\ 1,\ 10$ in each l case (see the plots). As a matter of fact, the integrals $I_{1,l}$ can be calculated analytically using the recurrence formula $\int \frac{x^m}{(1+x^2)^n} = \int \frac{x^{m-2}}{(1+x^2)^{n-1}} - \int \frac{x^{m-2}}{(1+x^2)^n}$, $m = 2(l+1)$ and $n = 2l+1$. The effect is stable and diminishes substantialy when one goes from one l to the next $l+1$ one. It starts to reveal itself at about half of the radius R within the focusing structure, reaches a maximum at $\leq \sqrt{2}R$ outside the lens which is moving toward the surface at higher l, and is insignificant beyond about four Rs. These strictly isospectral effects may show up in any focusing structure, for example in radar applications of the MFE lens. The isospectral parameter λ is controling the scale between the isospectral impedance and the normal one. This is why, we have chosen four scales corresponding to Z_{iso} one hundred times smaller than $Z_{1,l}$, ten times smaller, equal, and ten times bigger than $Z_{1,l}$ for $\lambda = 0.01$, 0.1, 1, and 10, respectively. The relative importance of the strictly isospectral effects depends on the value of λ. The case $l=0$ is special in the sense that $\mathcal{R}_0(\rho) = \rho - \mathrm{arctg}\rho + \lambda$ is a steadily increasing function and therefore does not show a peak structure as the other partial waves do. One can calculate easily the position of isospectral impedance peak to be $\rho_M = \sqrt{(1+\frac{1}{l})}$, which in the limit of high l almost reaches the surface

of the MFE lens, and also the velocity of this peak structure in the l- space $v_M = -\frac{1}{2l\sqrt{l(l+1)}}$, showing that in the limit of high l the peak is almost static.

(ii) Orbital capacity: Suppose we adopt a thermodynamic view consisting in considering the "fermionic" (scattering) effective potential $U^+(l)$ as a kind of metastable free energy and thus we introduce the DO "orbital capacity" $C_l^+ = \frac{dU^+}{dl}$, where $U^+(l) = l(l-1)/\rho^2 - (2l+1)(2l-2\kappa-1)/[\rho^{2(1-\kappa)}(1+\rho^{2\kappa})^2] + 2(2l+1)/[\rho^2(1+\rho^{2\kappa})^2]$. Inflection points of $U^+(l)$ imply peaks in C_l^+ and allow to identify the critical angular momenta beyond which some "scattering phase transitions" (quasi-bound states) would occur. As a matter of fact, this was the standpoint in [2] leading to DO quasi-bound states starting with the critical angular momentum for the MFE lens $l_{cr} = 6.876$.

(iii) Closeness to reflectionlessness: Finally, let us mention that if one uses the Langer change of variable ($\rho = e^x$) and function ($\psi = e^{x/2}\phi$) in the MFE nodeless sector [3], the MFE focusing potential turns into the following one $V_1(x) = -(n-1/2)(n+1/2)/\cosh^2 x$. Thus, the MFE case, and in general the DO class of focusing potentials are not reflectionless, a property requiring $n(n+1)$ in the numerator [5]. However, from their form, one can claim that the DO potentials are among the closest to the reflectionless ones. They miss that quality by a "semiclassical" 1/2 contribution.

To this end, we hint on the possible connection of the isospectral effect with the morphology-dependent resonances of the MFE lens [6] and also on the possibility of an experimental search of the effect since spherical gradient-index sphere lenses are already a technological reality [7].

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This work was partially supported by the CONACyT Projects 4868-E9406 and 3898P-E9608 and by PROMEP/97.

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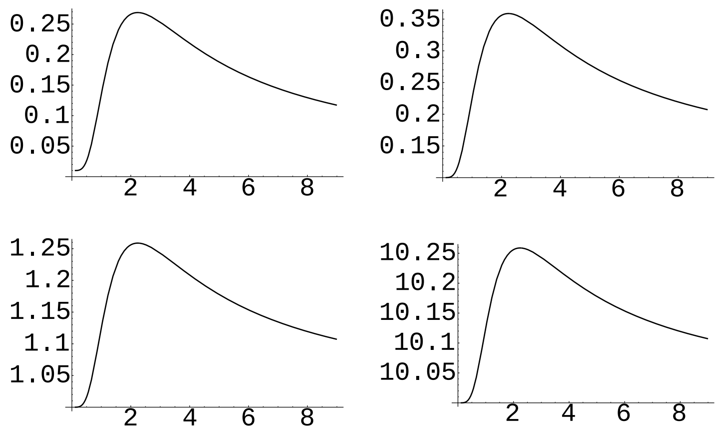

Fig. 1

$\mathcal{R}_1(\rho)$ for $\lambda = 0.01,\ 0.1,\ 1,\ 10$

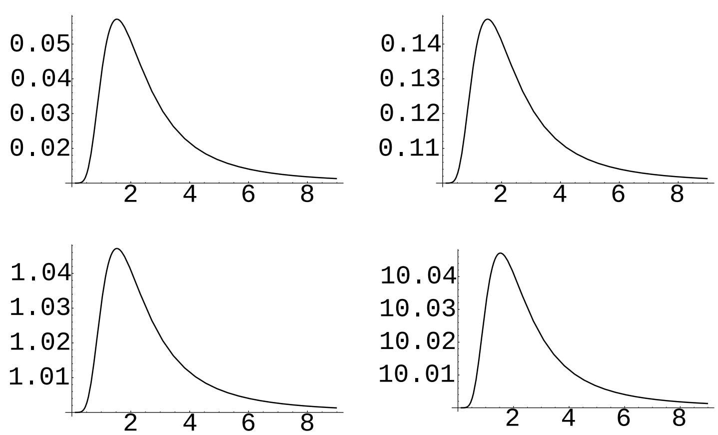

Fig. 2

$\mathcal{R}_2(\rho)$ for the same λ values.

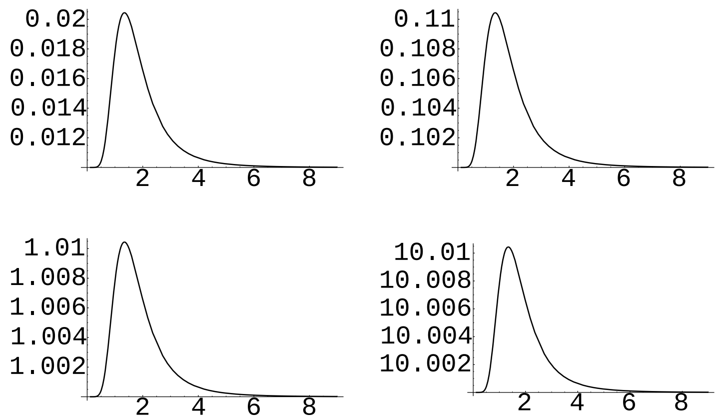

Fig. 3

$\mathcal{R}_3(\rho)$ for the same λ values.